The Category of Lagrangian Relations

Durgesh Kumar

University of Calgary

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Overview

- Motivating study of Lagrangian relations from Stabilizer theory
- Proving that Lagrangains compose, mathematically
- Proving that Lagrangians compose, graphically

Motivation: Stabilizer theory

- For p prime, define one qudit operators

$$X = \sum_{j=o}^{p-1} |j\rangle \langle j \oplus 1|$$

and

$$Z = \sum_{j=o}^{p-1} e^{\frac{i2\pi}{p}} |j\rangle \langle j|$$

- Operators of the form $X^q Z^r$, where $q, r \in \mathbb{Z}/p\mathbb{Z}$ are called **Weyl − Heisenberg Operators**.
- Extend it to n qubits by taking tensor products.

Stabilizer theory

- We have that $X^{\mathbf{Q}}Z^{\mathbf{R}}X^{\mathbf{Q}'}Z^{\mathbf{R}'} = (e^{\frac{i2\pi}{p}})^{\mathbf{Q}\cdot\mathbf{R}'-\mathbf{Q}'\cdot\mathbf{R}}X^{\mathbf{Q}'}Z^{\mathbf{R}'}X^{\mathbf{Q}}Z^{\mathbf{R}}$, where the inner product is taken mod p.
- Linear automorphisms of $(\mathbb{Z}/p\mathbb{Z})^{2n}$ which preserves this symplectic inner product is called a symplectomorphism.
- Corresponding to each symplectomorphism, there is unitary which maps the corresponding W.H. operators. We call these unitaries **Clifford Unitaries**.
- Stabilizer states are defined to be those states which can be prepared by applying an n-qubit Clifford Unitary to $|0\rangle^{\otimes n}$.

Prelude: Linear relations

The category of linear relations has:

- Objects: Finite-Dimensional Vector Spaces X, Y, Z...
- Morphisms: $R : X \to Y$ is a linear subspace of $X \oplus Y$
- Composition: Relational composition
- $\mathbb{I}_X = \Delta_X$

Symplectic Vector Space

Definition

A symplectic vector space $V_{\mathbb{F}}$ is a tuple $(V_{\mathbb{F}}, \Omega)$ such that $\Omega : V \times V \to \mathbb{F}$ is

- Bilinear; linear in each argument separately
- Alternating; $\Omega(v, v) = 0, \forall v \in V$
- Non-degenrate; $\Omega(u, v) = 0, \forall v \in V$ implies $u = 0$

Symplectic Vector Space

- $W^{\Omega} := \{v \in V : \Omega(v, w) = 0, \forall w \in W\}$
- $dim(W) + dim(W^{\Omega}) = dim(V)$
- A symplectomorphism between symplectic vector spaces is a linear isomorphism that also preserves the symplectic form.

Important subspaces

A subspace $W \subseteq V$ is called

- Isotropic if $W \subset W^{\Omega}$
- Co-Isotropic if $W^{\Omega} \subset W$
- Lagrangian if $W = W^{\Omega}$
- Symplectic if $W \cap W^{\Omega} = \{0\}$

Symplectic form on direct sum

- If (V, Ω) *and* $(V', \Omega^{'})$ are symplectic vector space, then $V \oplus V'$ is a symplectic vector space with the symplectic form $\Omega_{V \oplus V'}((v, v'), (w, w')) := \Omega_V(v, v) + \Omega_W(w, w')$
- We will denote by $\bar{V}$, the vector space V with the sympelctic form given by $-\Omega_V$

The Category of Lagrangian relations (?)

The category of Lagrangian Relations has:

- Objects: Symplectic Vector Spaces
- Morphisms: $R : X \to Y$ is a lagrangian subspace of $X \oplus \bar{Y}$
- Composition: Relational composition
- $\mathbb{I}_X = \Delta_X$

Reduction

Theorem

Let W be any subspace of V. Then $W/W \cap W^{\Omega}$ has natural symplectic structure given by $[\Omega]([u],[v]) := \Omega(u,v) \forall u, v \in W$

Co-Isotropic Reduction

Theorem

Let W be a co-isotropic subspace of V. If $L \subset V$ is an (co)-isotropic subspace, then $\rho(L \cap W)$ is a (co)-isotropic subspace of $W/W \cap W^{\Omega}$.

Composition of Lagrangians

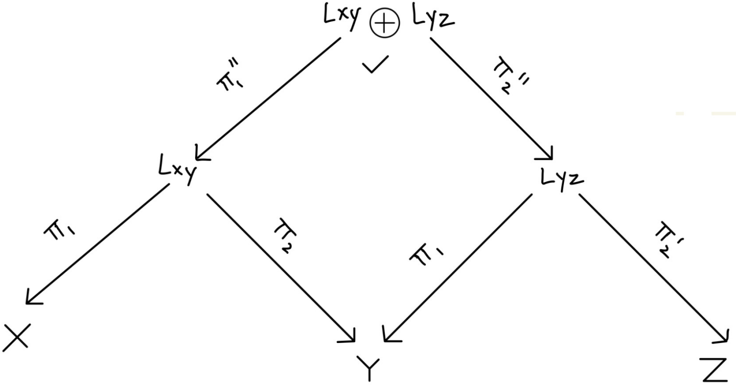

Figure: Composition of Lagrangians

Composition of Lagrangians

- $L_{XY} \oplus_Y L_{YZ} = \{((x, y), (y^{'}, z)) \in L_{XY} \oplus L_{YZ} : y = y^{'}\} = (L_{XY} \oplus L_{YZ}) \cap (X \oplus \Delta_Y \oplus Z)$
- And now consider,
  $f = (\pi_1^{L_{XZ}} \pi_1^{L_{XY}}, \pi_2^{L_{XZ}} \pi_2^{L_{YZ}}) : L_{XY} \oplus_Y L_{YZ} \to X \oplus Z$
  Then, $L_{XY} L_{YZ} = Im(f)$

The Category of Lagrangian relations (✓)

The category of Lagrangian Relations has:

- Objects: Symplectic Vector Spaces
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- Composition: Relational composition
- $\mathbb{I}_X = \Delta_X$

The Prop of Lagrangian Relations

Definition

Given a field k, the prop of Linear Relations over k, $\mathbf{LinRel_k}$ has morphisms $n \to m$ as linear subspaces of $k^n \oplus k^m$. Composition is given by relational composition and tensor is given by the direct sum.

LinRel$_k$

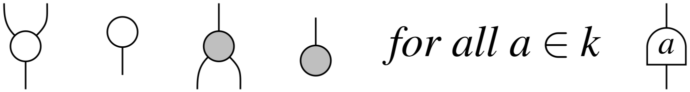

Figure: Generators for the Prop of Matrices, cb$_k$

- Generators for **LinRel$_\mathbf{k}$** is given by $(cb_k)^{op} + cb_k$

Premelinaries to define LangRel_k

Theorem

Any symplectic vector space (V, Ω) *of dimension 2n, has a basis* $\{e_1, e_2, ...e_n, f_1, f_2, ...f_n\}$ *such that* $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$ *and* $\Omega(e_i, f_j) = \delta_{ij}$

Premelinaries

Theorem

Under such a basis, Ω has the following form:

$$\Omega(u, v) = (-u-) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} (-v^T-)$$

Theorem

Every finite dimensional vector space of dimension 2n is symplectomorphic to a vector space with the symplectic form given above.

The Prop of Lagrangian relations

Definition

Given a field k, the prop of Lagrangian Relations, $\mathbf{LagRel_k}$ has morphisms $k^{2n} \to k^{2m}$ as Lagrangian subspaces of the sympletic vector space $k^{n+m} \oplus k^{n+m}$ with the standard symplectic form. Composition is given by relational composition and the tensor is given by the graded direct sum.

Graded sum

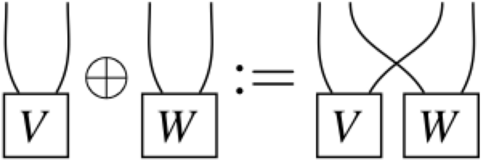

Figure: Tensor product in LagRel_k

LagRel_k

Theorem

The forgetful functor $U : LagRel_k \to LinRel_k$ is faithful, strong symmetric monoidal.

Lag in terms of Rel

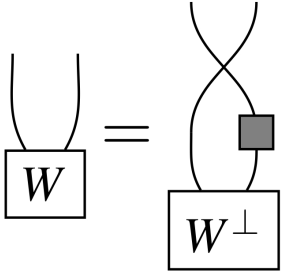

Figure: LagRel$_k$ *in terms of LinRel$_k$*

Lagrangians Compose

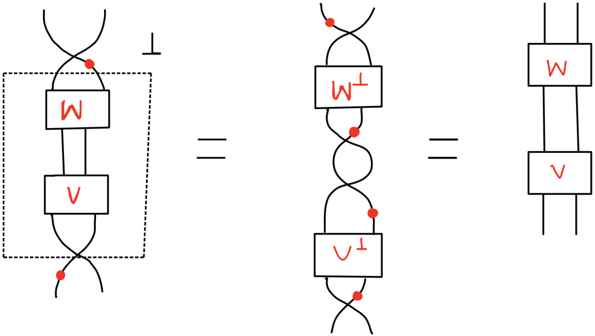

Figure: Composition of Lagrangians

Lagrangians Compose

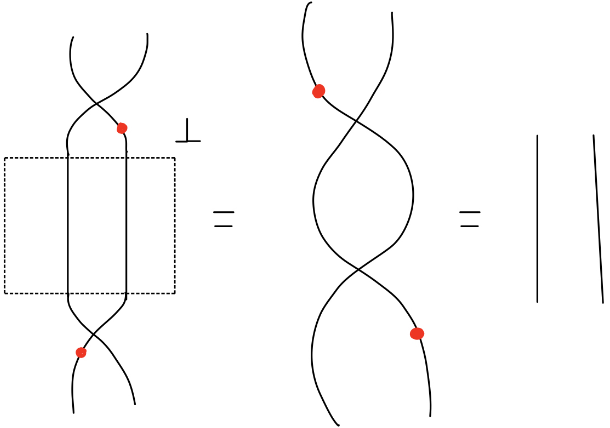

Figure: Composition of Lagrangians

LagRel_k

Theorem

There is a faithful, strong symmetric monoidal functor L : $LinRel_k \to LagRel_k$ given by the following

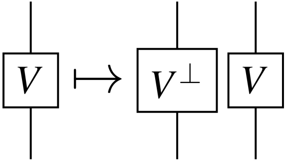

Figure: *Functor from $LagRel_k$ to $LinRel_k$*

Gneerators for LagRel$_k$

Theorem

For any field k, the maps in **LagRel$_\mathbf{k}$** *is generated by the maps in $L(LinRel_k)$ and a family of discard morphisms for each $a \in k$*

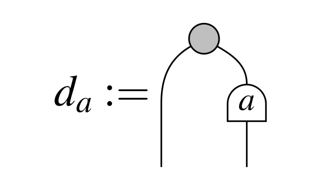

Figure: *Discard morphism*

Conclusion

- There are good reasons for studying the category of Lagrangians relations.
- We can prove that Lagrangian relations compose mathematically, but it requires some bit of mathematical machinery.
- Working with the Darboux basis, we can use the graphical calculus of linear relations to show how Lagrangians look like and that they compose.

References

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